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# ON A METHOD OF OBTAINING SPECTRAL RELATIONSHIPS FOR INTEGRAL OPERATORS OF MIXED PROBLEMS OF MECHANICS OF CONTINUOUS MEDIA*

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The method of orthogonal polynomials, and its generalization, the method of orthogonal functions /1,2/ applied for the investigation of complex mixed problems of the mechanics of continuous media, are based on the utilization of spectral relationships that invert the main (singular) part of the kernel of the integral equation of the problem under consideration. A sufficiently general approach to the derivation of spectral relationships that is based on potential theory is proposed. Eigenfunctions are obtained in the problem of impressing a strip stamp in an elastic half-space, as are also the odd eigenfunctions of a logarithmic series in the case of two symmetric intervals. An an application of the results obtained, the solution is constructed for any value of a certain dimensionless parameter, for the plane contact problem of the impression of a rigid stamp into the surface of an elastic strip which is under an interlayer of the type of a covering resting on an undeformable foundation.

1. We consider the three-dimensional contact problem of frictionless impression of a rigid stamp in the surface of an elastic $(G, \nu)$ half-space occupying the domain $-\infty < x, y < \infty$, $z > 0$.

As is known /3/, the integral equation of such a problem has the form

$$\iint_\Omega \frac{p(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} = 2\pi\theta\,[\delta - \alpha x + \beta y - f(x, y)] \tag{1.1}$$

$$(x, y) \in \Omega, \quad \theta = G(1-\nu)^{-1}$$

Here $p(x, y)$ are the normal stresses that are unknown under the stamp, $\Omega$ is the contact domain between the stamp and the surface of the half-space, $f(x, y)$ is a function describing the shape of the stamp base, $\delta - \alpha x + \beta y$ is its rigid displacement under the action of a force $P$ and moments $M_x, M_y$.

Equation (1.1) is valid under the evident statics conditions

$$P = \iint_\Omega p(x, y)\, dx\, dy \tag{1.2}$$

$$M_x = \iint_\Omega y p(x, y)\, dx\, dy, \qquad M_y = \iint_\Omega x p(x, y)\, dx\, dy$$

expressing the equilibrium of the stamp on the half-space surface and being used to determine the relationship between $P, M_x, M_y$ and $\delta, \alpha, \beta$.

We introduce the simple layer potential of density $p(x, y)$ distributed over the plane domain $\Omega$

$$\omega(x, y, z) = \iint_\Omega \frac{p(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}} \tag{1.3}$$

It has been shown /3/ that $\omega(x, y, z)$ is a harmonic function everywhere in space exept on a plane slit in the domain $\Omega$, and it vanishes at infinity as $PR^{-1}$ $(R = \sqrt{x^2 + y^2 + z^2})$. Moreover, the function $\omega(x, y, z)$ is continuous in all space, including the domain $\Omega$ also, and its normal derivative undergoes discontinuity in going from one side of the slit to the other, namely

$$\left(\frac{\partial \omega}{\partial z}\right)_{z \to \pm 0} = \begin{cases} \mp 2\pi p(x, y), & (x, y) \in \Omega \\ 0, & (x, y) \overline{\in} \Omega \end{cases} \tag{1.4}$$

We set the following boundary condition on the slit

$$\omega(x, y, 0) = 2\pi\theta\,[\delta - \alpha x + \beta y - f(x, y)], \quad (x, y) \in \Omega \tag{1.5}$$

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Then the solution of the integral equation (1.1) is equivalent to the problem (1.3)−(1.5) of determining the harmonic function $\omega(x, y, z)$.

The general idea of the method of solving the Dirichlet problem (1.3), (1.5) in which the unknown value of the density $p(x, y)$ (1.4) is determined and elucidated in /3/. We demonstrate it in two important particular examples.

It is known /4/ that the contact problem concerning the frictionless impression of a stamp of strip planform in an elastic half-space in the case when the seat beneath it is determined by the formula $w(x, y) = g(\lambda, x)\cos\lambda y$ reduces to the integral equation

$$\int_{-a}^{a} \varphi(\lambda, \xi) K_0(\lambda|\xi - x|)\, d\xi = \pi\theta g(\lambda, x) \quad (|x| \leqslant a) \tag{1.6}$$

$$p(x, y) = \varphi(\lambda, x)\cos\lambda y$$

Here $2a$ is the width of the strip, $K_0(t)$ is the Macdonald function, and $\lambda$ is an arbitrary positive number.

We introduce the potential

$$\omega(\lambda, x, z) = \frac{1}{2\pi\theta}\int_{-a}^{a} \varphi(\lambda, \xi) K_0[\lambda\sqrt{(x-\xi)^2 + z^2}]\, d\xi \tag{1.7}$$

Then the solution of the integral equation (1.6) can be found, as mentioned above, by solving the following Dirichlet boundary value problem /4/:

$$\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial z^2} - \lambda^2\omega = 0 \quad (x \in [-a, a], z \neq 0) \tag{1.8}$$

$$\omega(\lambda, x, 0) = g(\lambda, x)\ (|x| \leqslant a), \ \omega(\lambda, x, z) \to 0, \ (x^2 + z^2) \to \infty$$

The formulas (1.4) are here rewritten in the form

$$\frac{\partial\omega(\lambda, x, \pm 0)}{\partial z} = \mp\frac{1}{\pi\theta}\varphi(\lambda, x) \quad (|x| < a) \tag{1.9}$$

$$\frac{\partial\omega(\lambda, x, 0)}{\partial z} = 0 \quad (|x| > a)$$

To construct the solution of the problem (1.8), we go over to the elliptic coordinates

$$x = a\cos\eta\,\mathrm{ch}\,\xi, \quad z = a\sin\eta\,\mathrm{sh}\,\xi$$

Then

$$\frac{\partial^2\omega}{\partial\xi^2} + \frac{\partial^2\omega}{\partial\eta^2} - \frac{a^2\lambda^2}{2}(\mathrm{ch}\,2\xi - \cos 2\eta)\,\omega = 0 \tag{1.10}$$

$$\omega(\lambda, 0, \eta) = g(\lambda, a\cos\eta), \quad \omega(\lambda, \infty, \eta) = 0$$

and we obtain in place of the first formula in (1.9)

$$\varphi(\lambda, a\cos\eta) = \frac{\theta}{a|\sin\eta|}\frac{\partial\omega}{\partial\xi}\Big|_{\xi=0} \tag{1.11}$$

which is valid for all values of $\eta$.

Assuming the function $g(\lambda, x)$ to be such that it can be expanded into a uniformly convergent series of periodic Mathieu functions in the interval $-a \leqslant x \leqslant a$:

$$g(\lambda, a\cos\eta) = \sum_{n=0}^{\infty} h_n\,\mathrm{ce}_n(\eta, -q) \quad \left(q = \frac{\lambda^2 a^2}{4}\right) \tag{1.12}$$

we will seek the solution of the problem (1.10) in the form

$$\omega(\lambda, \xi, \eta) = U(\xi)\,V(\eta) \tag{1.13}$$

We arrive at the necessity to study the Mathieu equations

$$V'' + (\alpha + 2q\cos 2\eta)\,V = 0, \ U'' - (\alpha + 2q\,\mathrm{ch}\,2\xi)\,U = 0 \tag{1.14}$$

$$U(0)\,V(\eta) = g(\lambda, a\cos\eta), \ U(\infty) = 0 \quad (\alpha = \mathrm{const})$$

Furthermore, following the general theory of the Mathieu equation /5/, we write the general solution (1.13) of the boundary value problem (1.10) formulated that satisfies the second condition of (1.14), in the form

$$\omega(\lambda, \xi, \eta) = \sum_{n=0}^{\infty} A_n\,\mathrm{Fek}_n(\xi, -q)\,\mathrm{ce}_n(\eta, -q) \tag{1.15}$$

Now utilizing the first boundary condition (1.14) as well as the representation (1.12), we obtain an expression for the arbitrary constants $A_n$ in (1.15), and taking account of (1.11) we find

$$\varphi(\lambda, a\cos\eta) = \frac{\theta}{a\,|\sin\eta|}\sum_{n=0}^{\infty} h_n \frac{\mathrm{Fek}_n'(0,-q)}{\mathrm{Fek}_n(0,-q)}\,\mathrm{ce}_n(\eta,-q) \tag{1.16}$$

Since any harmonic in (1.16) satisfies (1.10), then by substituting it into the integral equation (1.6), in conformity with (1.12) we arrive at the following spectral relationship for the Mathieu functions:

$$\int_{-a}^{a} \frac{\mathrm{ce}_n(\arccos \xi a^{-1}, -q)}{\sqrt{a^2-\xi^2}} K_0(\lambda\,|\xi - x|)\,d\xi = \frac{\pi\,\mathrm{Fek}_n(0,-q)}{\mathrm{Fek}_n'(0,-q)}\,\mathrm{ce}_n(\arccos x a^{-1}, -q) \tag{1.17}$$

We note that the latter has been obtained implicitly in /4/.

We consider the odd plane contact problem of the impression of two symmetrically arranged stamps in an elastic half-plane. The integral equation of such a problem has the form /1/

$$\int_{b}^{a} \varphi(\xi)\ln\left|\frac{\xi+x}{\xi-x}\right| d\xi = f(x) \quad (b \leqslant x \leqslant a) \tag{1.18}$$

$$M = 2\int_{b}^{a} \varphi(x)\,x\,dx$$

We introduce the logarithmic potential

$$\omega(x, y) = \int_{L} \ln\frac{1}{\sqrt{(x-\xi)^2+y^2}}\,\varphi(\xi)\,d\xi, \quad L = \{x : b \leqslant |x| \leqslant a\} \tag{1.19}$$

Then the solution of the integral equation (1.18) is equivalent to the solution of the following external Dirichlet boundary value problem

$$\Delta\omega = 0, \quad (x, y) \,\overline{\in}\, L, \ x^2 + y^2 \neq 0 \tag{1.20}$$
$$\omega(x, y) = f(x),\ y = 0; \quad \omega(x, y) \to 0,\ (x^2 + y^2) \to \infty$$

After the function $\omega(x, y)$ in (1.19) has been constructed, the density of the potential $\varphi(x)$ will be determined from the formula (compare with (1.14))

$$\varphi(x) = -\frac{1}{\pi}\lim_{y\to+0}\frac{\partial\omega(x,y)}{\partial y} \quad (b \leqslant |x| \leqslant a) \tag{1.21}$$

We construct the solution of the problem (1.20) by the method of conformal mapping. To this end we note that the function /6/

$$z = b\,\mathrm{sn}\,[\pi^{-1}K'(k)\ln\zeta,\ k] \quad (k = b/a) \tag{1.22}$$

maps the complex plane $z = x + iy$ slit along $L$ into the circular ring

$$q_0 \leqslant \rho \leqslant q_0^{-1}, \ q_0 = \exp\,[-\pi K(k)/K'(k)] \tag{1.23}$$

of the complex plane $\zeta = \rho e^{i\psi} = \xi + i\eta$. Here $\mathrm{sn}(u, k)$ is the Jacobi elliptic function $K(k), K'(k)$ are complete elliptic integrals of the first kind of argument $k$ and $k' = \sqrt{1-k^2}$, respectively.

In the conformal mapping (1.22) the upper half-plane $\mathrm{Im}\, z > 0$ is mapped the upper half ring $\{q_0 < \rho < q_0^{-1},\ 0 < \psi < \pi\}$ the lower half-plane $\mathrm{Im}\, z < 0$ into the lower half ring $\{q_0 < \rho < q_0^{-1},\ -\pi < \psi < 0\}$, and the infinitely remote point of the plane $z$ goes over into the point $\zeta = -1$ of the plane $\zeta$. In addition, the upper edge of the slit along the segment $[b, a]$ goes over into the upper half ring of the outer circle $\rho = q_0^{-1}$ of the ring, and the lower edge into the lower half ring of this same circle. Analogously, the upper edge of the slit along the segment $[-a\ -b]$ goes over into the upper semicircle of the inner circumference $\rho = q_0$ of the ring, and the lower edge goes over into the lower semicircle of this circumference.

Let us introduce the function

$$w = u + iv = \pi^{-1}K'(k)\ln\zeta \tag{1.24}$$

mapping the rectangle $\{-K(k) \leqslant u\ K(k),\ -K'(k) \leqslant v \leqslant K'(k)\}$ onto the above-mentioned circular ring. Separating real and imaginary parts in (1.24), we obtain

$$u = \pi^{-1}K'(k)\ln\rho, \quad v = \pi^{-1}K(k)\,\psi \quad (q_0 \leqslant \rho \leqslant q_0^{-1},\ -\pi < \psi \leqslant \pi) \tag{1.25}$$

From (1.23) - (1.25) we will have

$$z = b\,\mathrm{sn}(u + iv, k),\ x = b\,\mathrm{sn}(u, k)\,\mathrm{cn}(iv, k)\,\varkappa \tag{1.26}$$
$$y = -ib\,\mathrm{cn}(u, k)\,\mathrm{sn}(iv, k)\,\varkappa,\ \varkappa = \mathrm{dn}(iv, k)\,[1 - k^2\,\mathrm{sn}^2(u, k)\,\mathrm{sn}^2(iv, k)]^{-1}$$

where $\mathrm{cn}(u, k)$ and $\mathrm{dn}(u, k)$ are Jacobi elliptic functions.

The segment $[-a, -b]$ covered twice, or according to the above, the inner circumference $\rho = q_0$ of the ring corresponds to the coordinate line $u = -K(k)$, and the twice-covered segment $[b, a]$ or the outer circumference $\rho = q_0^{-1}$ of the ring corresponds to the coordinate line $u = K(k)$. Since $\mathrm{sn}[K(k), k] = 1$, then we find on the coordinate line $u = K(k)$ from the second formula in (1.26) by utilization of transformation formulas for Jacobi elliptic functions with imaginary argument /7/

$$x = b\,\mathrm{dn}^{-1}(v, k')\ (|v| \leqslant K'(k)) \tag{1.27}$$

On the other hand, by taking into account the representation of the function $\mathrm{dn}(v, k')$ by using an elliptic integral /7/, we find after simple operations

$$v = \int_1^{x/b} \frac{dt}{\sqrt{(t^2-1)(1-k'^2t^2)}} \quad (b \leqslant x \leqslant a) \tag{1.28}$$

For $-a \leqslant x \leqslant -b$ formula (1.28) should be continued oddly.

It follows from (1.28) that when $x$ grows from $b$ to $a$, then $v$ grows from 0 to $K'(k)$, and therefore (1.28) yields the dependence between the variables $v$ and $x$.

Furthermore, starting from (1.25) and (1.27), we set

$$f_1(\psi) = -f\left\{\frac{b}{\mathrm{dn}[\pi^{-1}K'(k)\psi, k']}\right\},\quad f_2(\psi) = -f_1(\psi)\quad (-\pi < \psi \leqslant \pi)$$

The function $f_1(\psi)$ is evidently defined on the inner circumference $\rho = q_0$ of the ring, and the function $f_2(\psi)$ on the outer circumference $\rho = q_0^{-1}$ of the same ring. Both are even functions of the variable $\psi$.

Now, the boundary value problem (1.20) for the plane $z$ with slit along $L$ goes over into the following boundary value problem for the circular ring in the plane $\zeta$ after the conformal mapping (1.22):

$$\frac{\partial^2 W}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial W}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial\psi^2} = 0 \tag{1.29}$$
$$W(\rho, \psi) = f_1(\psi),\ \rho = q_0;\quad W(\rho, \psi) = f_2(\psi),\ \rho = q_0^{-1}\,(-\pi < \psi \leqslant \pi)$$
$$W(1, \pm\pi) = 0$$

where $W(\rho, \psi) = \omega(x, y)$, and the relation between the variables $\rho, \psi$ and $x, y$ is by means of (1.25) and (1.26). At the same time, omitting the calculations, we represent the formula to calculate the potential density (1.21) in the form

$$\varphi(x) = \frac{a}{K'(k)\sqrt{(a^2-x^2)(x^2-b^2)}}\left[\rho\frac{\partial W}{\partial\rho}\right]_{\rho=q_0^{-1}} \quad (b \leqslant x \leqslant a) \tag{1.30}$$

Utilizing the method of separation of variables to solve the problem (1.29), we obtain /8/

$$W(\rho, \psi) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{h_n(\rho^n - \rho^{-n})\cos n\psi}{\mathrm{sh}[\pi n K(k)/K'(k)]} + \frac{h_0}{\pi}\frac{K'(k)}{K(k)}\ln\rho \tag{1.31}$$
$$f_j(\psi) = (-1)^j\left[h_0 + \sum_{n=1}^{\infty} h_n\cos n\psi\right] \quad (j = 1, 2)$$
$$q_0 \leqslant \rho \leqslant q_0^{-1},\quad -\pi < \psi \leqslant \pi$$

Taking account of (1.31), we conclude that the density (1.30) corresponding to the potential $W(\rho, \psi)$ will be expressed by the formula

$$\varphi(x) = \frac{a}{K'(k)\sqrt{(a^2-x^2)(x^2-b^2)}}\left\{\sum_{n=1}^{\infty} n h_n\,\mathrm{cth}\left[\frac{\pi n K(k)}{K'(k)}\right] \times T_n(X)| + \frac{h_0}{\pi}\frac{K'(k)}{K(k)}\right\} \quad (b \leqslant x \leqslant a) \tag{1.32}$$

$$X=\cos\psi,\quad \psi=\frac{\pi}{K'(k)}\int_1^{x/b}\frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}}$$

where $T_n(t)$ are Chebyshev polynomials of the first kind.

Now, in (1.32) let

$$h_m=0\ (m\neq n),\ h_n=1\ (m,n=0,1,2,\ldots)$$

Then each of the expansions (1.31) will contain just one harmonic which satisfies the boundary value problem (1.29). Substituting it into the integral equation (1.18), we arrive at the following spectral relation:

$$\int_b^a \ln\left|\frac{\xi+x}{\xi-x}\right|\frac{T_n(Y)\,d\xi}{\sqrt{(a^2-\xi^2)(\xi^2-b^2)}}=\lambda_n T_n(X)\quad (n=0,1,\ldots) \tag{1.33}$$

$$Y=\cos\alpha,\ \alpha=\frac{\pi}{K'(k)}\int_1^{\xi/b}\frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}}\quad (b\leqslant\xi\leqslant a)$$

$$\lambda_n=\frac{K'(k)}{an}\,\mathrm{th}\left[\frac{\pi nK(k)}{K'(k)}\right],\quad \lambda_0=\frac{\pi}{a}K(k)$$

The eigenfunctions of a logarithmic series yield the relation (1.33) in the case of two symmetric intervals.

Now, setting in (1.33)

$$a=e^{\gamma},\ b=e^{-\gamma},\ \xi=e^{\gamma\tau},\ x=e^{\gamma t}\ (\gamma>0)$$

we obtain

$$-\int_{-1}^{1}\ln\left|\mathrm{th}\,\frac{\gamma(\tau-t)}{2}\right|\frac{T_n(s)\,d\tau}{\Delta(\tau)}=\mu_n T_n(r)\quad (n=0,1,2,\ldots) \tag{1.34}$$

$$\mu_0=\frac{\sqrt{2}\,\pi e^{-\gamma}}{\gamma}K(e^{-2\gamma}),$$

$$\mu_n=\frac{\sqrt{2}\,e^{-\gamma}}{n\gamma}K'(e^{-2\gamma})\,\mathrm{th}\left[\pi n\frac{K(e^{-2\gamma})}{K'(e^{-2\gamma})}\right]\quad (n=1,2,\ldots)$$

$$s=\cos\alpha(\tau),\quad r=\cos\alpha(t),\quad \Delta(\tau)=\sqrt{\mathrm{ch}\,2\gamma-\mathrm{ch}\,2\gamma\tau}$$

$$\alpha(\tau)=\frac{\pi}{K'(e^{-2\gamma})}F\left(\arcsin\sqrt{\frac{e^{2\gamma}-e^{-2\gamma\tau}}{2\,\mathrm{sh}\,2\gamma}},\quad \sqrt{1-e^{-4\gamma}}\right)$$

Here $F(x,y)$ is the elliptic integral of the first kind.

The eigenfunctions of the kernel $\ln|\mathrm{th}\,\gamma(t-\tau)2^{-1}|\ (-1\leqslant\tau,\ t\leqslant1)$ yield the spectral relationship (1.34).

2. We use the obtained spectral relationship (1.34) to solve integral equations of mixed problems of the mechanics of a continuous medium (the paper /2/ is devoted to the application of the spectral relation (1.17) to solve the integral equations of such problems).

It is known /9/ that a broad class of linear mixed problems of elasticity and viscoelasticity theory (contact problems), hydromechanics (linear problems of gliding, flow around thin profiles and surfaces, problems of linear supercavitation, etc.) in planar and three-dimensional formulations reduce to a convolution integral equation of the first kind in a finite interval

$$\int_{-1}^{1}\varphi(\xi)N\left(\frac{\xi-x}{\lambda}\right)d\xi=\pi f(x)\quad (|x|\leqslant1) \tag{2.1}$$

$$N(t)=\frac{1}{2}\int_{-\infty+ic}^{\infty+ic}L(u)u^{-1}e^{-iut}\,du\left(t=\frac{\xi-x}{\lambda},\ u=\sigma+i\tau,|c|<\gamma\right)$$

where the function $L(u)$ is regular in the strip $|\tau|\leqslant\gamma,\ |\sigma|<\infty$ and the following asymptotic formulas hold

$$L(u)=Au+O(|u|^3)\ (|u|\to0),\quad L(u)=\mathrm{sgn}\,\sigma+O(e^{-\nu|\sigma|})\quad (|\sigma|\to\infty) \tag{2.2}$$

The $A,\lambda,\gamma$ and $\nu$ in (2.1) and (2.2) are constants whose values are determined by specific problems Constraints imposed on the right side of equation (2.1) will be indicated below.

Because of condition (2.2), the function $L(u)$ can be represented in the form /9/

$$L(u) = \operatorname{th} Au + G(u)$$

from which

$$N(t) = -\ln\left|\operatorname{th}\frac{\pi t}{4A}\right| + H(t), \quad H(t) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{G(\sigma)}{\sigma}e^{-i\sigma t}\,d\sigma \tag{2.3}$$

Here $H(t)$ as a function of the complex variable $w = t + is$, is regular in the strip $|s| < \inf(\nu, 2A)$, $|t| < \infty$ and, moreover /9/

$$H(t) = O(e^{-\varkappa|t|})\ \left(|t| \to \infty,\ \varkappa = \inf\left(\gamma, \frac{\pi}{2A}\right)\right)$$

Therefore, the first component in the expression (2.3) for $N(t)$ reflects completely all the fundamental properties of the kernel of the integral equation (2.1) for all $t \in [0, \infty)$. The second component in (2.3) is an arbitrarily smooth function for $t \in [0, \infty)$ and plays the part of a small addition. Therefore, to construct a method of solving the integral equation (2.1) which is identically effective for all values of $\lambda \in (0, \infty)$ it is necessary to invert exactly the integral operator

$$L^*\varphi = -\int_{-1}^{1}\varphi(\xi)\ln\left|\operatorname{th}\frac{\pi(\xi - x)}{4A\lambda}\right|d\xi$$

The scheme of such a method is elucidated in /9/. In conformity with it, we assume that the even function $f(x) \in W^1_{4+0}(-1, 1)$ (definitions of all the functional spaces mentioned are given in /9/) and we represent $\varphi(x)$ in (2.1) in the form

$$\varphi(x) = \frac{\theta(x)}{\Delta(x)}, \quad \Delta(x) = \sqrt{\operatorname{ch} 2\gamma - \operatorname{ch} 2\gamma x}, \tag{2.4}$$

$$\gamma = \frac{\pi}{2A\lambda}, \quad \theta(x) \in L_2^{1/2}(-1, 1)$$

We then seek the function $\theta(x)$ in the relation (2.4) in the form of the following Fourier series in Chebyshev polynomials

$$\theta(x) = \sum_{j=0}^{\infty} a_j T_{2j}(r) \tag{2.5}$$

We expand the function $f(x)$ as well as the regular addition of the kernel $H(t)$ in a unitary and binary series, respectively, in the mentioned system of polynomials. We will have

$$f(x) = \sum_{n=0}^{\infty} f_n T_{2n}(r), \quad H(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e_{mn} T_{2m}(s)\, T_{2n}(r) \tag{2.6}$$

Because of the above-mentioned properties of the functions $f(x)$ and $H(t)$ the series (2.6) converge uniformly to them for all $|x| \leqslant 1$, $|\xi| \leqslant 1$, $\lambda > 0$.

Using the orthogonality condition of the Chebyshev polynomials

$$\int_{-1}^{1}\frac{T_n(r)\,T_m(r)\,dx}{\Delta(x)} = \frac{\sqrt{2}}{\gamma e^{\gamma}}K'(e^{-2\gamma})\begin{cases}1 & (m = n = 0)\\ 1/2 & (m = n \neq 0)\\ 0 & (m \neq n)\end{cases} \tag{2.7}$$

we obtain

$$f_n = \beta_n \int_{-1}^{1} f(x)\frac{T_{2n}(r)\,dx}{\Delta(x)} \tag{2.8}$$

$$e_{mn}(\lambda) = \beta_m\beta_n\int_{-1}^{1}\int_{-1}^{1} H(t)\,\frac{T_{2m}(s)\,T_{2n}(r)\,d\xi\,dx}{\Delta(\xi)\,\Delta(x)}$$

$$\beta_0 = \gamma e^{\gamma}[\sqrt{2}\,K'(e^{-2\gamma})]^{-1}, \quad \beta_n = 2\beta_0 \quad (n \geqslant 1)$$

Now, substituting (2.3)—(2.6) in the integral equation (2.1), using the relations (1.34) and (2.7), equating coefficients of the left and right sides in Chebyshev polynomials of identical number in the expression obtained, we arrive at the following infinite system to determine the unknown coefficients $a_j$:

$$a_j + \sum_{n=0}^{\infty} c_{nj}(\lambda)\, a_n = b_j \quad (j = 0, 1, 2, \ldots) \tag{2.9}$$
$$b_j = \pi \mu_j^{-1} f_j, \; c_{nj}(\lambda) = \mu_j^{-1} \beta_n^{-1} e_{nj}(\lambda)$$

Having obtained estimates of the type in /9/ for the coefficients of the system (2.9), it can be asserted that the infinite system (2.9) is quasi-completely regular for $\lambda > 0$. Moreover, a certain $\lambda_0 > 0$ can be mentioned such that the infinite system (2.9) will be completely regular /10/ for $\lambda > \lambda_0$

Having solved the system (2.9), we then find the function $\varphi(x)$, the solution of the integral equation (2.1) for any value of the parameter $\lambda \in (0, \infty)$, by means of (2.4) and (2.5).

3. As an illustration, we consider the following contact problem. Let a rigid stamp of width $2a$ with a flat base interact with the surface of an elastic strip of thickness $H$ under which lies an interlayer of the covering type $(-h \leqslant y \leqslant 0,\, h \ll H)$ lying without friction on an undeformable base (Fig.1). The condition of rigid adhesion is realized between the strip and the covering. The stamp is impressed in the strip by a force $P$. The friction forces on the contact line are assumed absent. The strip is not loaded outside the stamp, and we neglect mass forces. The case of the plane state of strain is examined.

Under the assumptions made the boundary conditions of the problem will have the form

$$y = H : \tau_{xy}^{-1} = 0, \quad \sigma_y^1 = 0 \; (|x| > a), \quad v_1 = \delta \; (|x| \leqslant a) \tag{3.1}$$
$$y = 0: u_1 = u_2, \; v_1 = 0, \; \sigma_y^1 = \sigma_y^2, \; \tau_{xy}^1 = \tau_{xy}^2$$
$$2G_2 h u_1'' = -(1 - \nu_2)\, \tau_{xy}^1 - \nu_2 h\, (\sigma_y^1)'$$

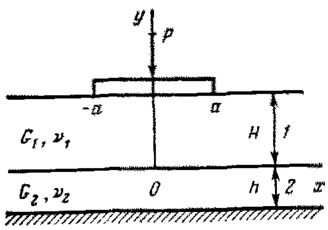


Fig.1

The last boundary condition in (3.1) is brought together with the middle plane of the covering on the lower boundary of the strip $y = 0$ because of the slightness of the thickness of the interlayer $(h \ll H)$.

Applying the Fourier integral transform in the variable $x$ /11/ to solve the problem formulated, and going over to the dimensionless variables and notation

$$x' = xa^{-1}, \; \xi' = \xi a^{-1}, \; \lambda = Ha^{-1}, \; \delta' = \delta a^{-1}, \; q(x)\, \theta^{-1} = \varphi(x'),$$
$$\theta = G_1 (1 - \nu_1)^{-1}$$

(we later omit the primes), we obtain the integral equation (2.1) for the unknown contact pressures $q(x)$ under the stamp, where

$$L(u) = [\,(2\varkappa_1 \operatorname{sh} 2u - 4u)\, u + 4m\,(nu \operatorname{sh} 2u + \operatorname{ch} 2u - 1)] \times \tag{3.2}$$
$$[(2\varkappa_1 \operatorname{ch} 2u + \varkappa_1^2 + 1 + 4u^2)\, u + 4m\,(n\varkappa_1 u + nu \operatorname{ch} 2u + 2u + \operatorname{sh} 2u)]^{-1}$$
$$m = \frac{G_1 H (1 - \nu_1)(1 - \nu_3)}{h (G_2 - \nu_2 G_1)}, \; n = \frac{h \nu_2}{H (1 - \nu_2)}, \quad \varkappa_1 = 3 - 4\nu_1$$
$$A = 4\, \frac{\varkappa_1 - 1 + 2m\,(1 + n)}{(\varkappa_1 + 1)^2 + 4m\,[4 + n\,(1 + \varkappa_1)]}, \quad G_2 > \nu_2 G_1$$

Let us note that the statics condition

$$Q = P\,(a\theta)^{-1} = \int_{-1}^{1} \varphi(x)\, dx$$

expressing the equilibrium of the stamp on the upper face of the strip, must be appended to (2.1).

The approximate solution of the integral equation (2.1), (3.2) can be obtained by the method elucidated in Sect.2:

Table 1

| $\lambda$ | $x = 0{,}0$ | 0.2 | 4 | 0.6 | 0.8 | 0.95 | $Q\delta^{-1}$ |
|---|---|---|---|---|---|---|---|
| $1/2$ | 3.905 | 3.887 | 3.863 | 3.821 | 3.961 | 5.739 | 8.571 |
| 1 | 1.964 | 1.967 | 1.984 | 2.048 | 2.328 | 3.804 | 4.648 |
| 2 | 1.011 | 1.021 | 1.058 | 1.150 | 1.425 | 2.555 | 2.716 |

We set $m = 10$, $n = 0.1$, $\nu_1 = 0.3$ in (3.2). The distribution of the unknown contact pressures $\varphi(x)\, \delta^{-1}$ under the stamp as well as the values of the quantities $Q\delta^{-1}$ are given in the table for certain $\lambda$. It is important that the number of equations in the shortened system (2.9) does not exceed five for different values of the parameter $\lambda \in (0, \infty)$. The accuracy guaranteed for the solution of the problem is not less than 2%.

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